APPENDIX A:

A1. Proof of Lemma 1

First, suppose that $\phi = \overline{\phi}$, then token production will be equal to what is needed for a stable price $\Delta^* = \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. Since token production is equal to the amount of tokens lost it will be the case that $\gamma = 1$ and thus the price level will be constant over time. The price of tokens will be the same as the previous period $\phi_{+1} = \phi$. This will lead to the same amount of tokens being created as the previous period $\Delta^*_{+1} = \Delta^*$.

This process will repeat itself ensuring token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. Thus, $\gamma = 1$ for the remaining periods making a steady state possible.

Next, assume $\phi > \overline{\phi}$, then token production will greater than what is needed for a stable price $\Delta^* > \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. Since token production will be greater than the amount of tokens lost it will be the case that $\gamma > 1$ and thus inflation occurs. Over time, the price of tokens will be less than the previous period $\phi_{+1} < \phi$. This will lead to less tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} < \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with inflation in this economy.

Lastly, suppose $\phi < \overline{\phi}$, then token production is less than what is needed for a stable price $\Delta^* < \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. There will be less new tokens created than tokens lost from the money supply in the previous period $\Delta^* < \psi M_{-1}$. Since token production will be less than the amount of tokens lost it will be the case that $\gamma < 1$ and thus deflation is observed. The price of tokens will be greater than the previous period $\phi_{+1} > \phi$. This will lead to more tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} > \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with deflation in this economy. ■

A2. Proof of buyer's loan decision for government money w/ type I banks

The buyer's first order condition for l_g is

$$W_{m,g} + \lambda^g \leq -W_{l^g} + \lambda_l^g \tag{A1}$$

Identical to the loan decision of private money, the rate of currency loss will play no direct role in the loan decision.

From the previous envelope conditions: $W_{m,g} = \phi^g(1 + i^g)$ and $W_{l^g} = -\phi^g(1 + i_l^g)$ reduces down to

$$\lambda^g \leq \phi^g(i_l^g - i^g) \tag{A2}$$

Upon substituting for λ^g, the quantity traded in the first market satisfies:

$$\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = \frac{\phi^g(1 + i_l^g)}{\phi^g(1 + i^g)} + \frac{\lambda_l^g}{\phi^g(1 + i^g)} \tag{A3}$$

If the borrowing constraint does not bind,

$$\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = \frac{\phi^g(1 + i_l^g)}{\phi^g(1 + i^g)} \tag{A4}$$

■

A3. Proof of Type I Bank's Problem

We can represent the bank's problem as

$$\mathscr{L} \max_{l_g} = (i_l^g + f - (1 - f)i_d)l_g - \lambda_L(l - \bar{l}) + \lambda_\Gamma(u(q_b, q_b^g) - (1 + i_l^g)l\phi^g - \Gamma) \tag{A5}$$

where λ_L is the Lagrange multiplier on the constraint on a bank's lending. Also, λ_Γ is the

Lagrange multiplier on the participation constraint of the borrower. The bank's first order condition for how much to loan becomes:

$$i_l^g + f - (1-f)i_d - \lambda_L + \lambda_\Gamma \Big[u_{q_b^g}(q_b, q_b^g) \frac{dq_b^g}{dl_g} - (1+i_l^g)\phi^g \Big] = 0 \tag{A6}$$

Similar to what was described in BCW, with $i_l^g + f - (1-f)i_d^g > 0$ the bank will loan as much as possible. Therefore, $u_{q_b^g}(q_b, q_b^g)\frac{dq_b^g}{dl_g} = (1+i_l^g)\phi$ and it will be the case that $\lambda_\Gamma > 0$. However, competition will drive interest rates to parity: $i_l^g = (1-f)i_d^g - f$.

From $q_b^g = (1-\psi^g)(m_g + l_g)/p^g$

$$\frac{dq_b^g}{dl_g} = \frac{(1-\psi^g)}{p^g} \Rightarrow \frac{dl_g}{dq_b^g} = \frac{p^g}{(1-\psi^g)} \tag{A7}$$

As p^g increases, it costs more to obtain goods in the DM and the buyer will choose to borrow more funds. Since $p^g\phi^g(1+i^g) = c_{1,q_s^g}(q_s, q_s^g)$

$$\frac{dq_b}{dl_g} = \frac{(1+i^g)(1-\psi^g)\phi^g}{c_{1,q_s^g}(q_s, q_s^g)} \tag{A8}$$

Thus, a bank's decision on how much to lend becomes

$$i_l^g + f - (1-f)i_d^g - \lambda_L + \lambda_\Gamma \Big[u_{q_b^g}(q_b, q_b^g) \frac{(1+i^g)(1-\psi^g)\phi^g}{c_{1,q_s^g}(q_s, q_s^g)} - (1+i_l^g)\phi^g \Big] = 0 \tag{A9}$$

Since the bank will receive no profits due to free entry:

$$\phi^g \lambda_\Gamma \Big[\frac{(1+i^g)(1-\psi^g)u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} - (1+i_l^g) \Big] = \lambda_L \tag{A10}$$

$$\frac{(1+i^g)(1-\psi^g)u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} = 1 + i_l^g + \frac{\lambda_L}{\phi\lambda_\Gamma} \tag{A11}$$

■

A4. Proof of Type II Bank's Problem

We can represent the bank's problem for private money as

$$\mathscr{L} \max_{l} = (i_l + f - (1-f)i_d)l_g - \lambda_L(l - \bar{l}) + \lambda_\Gamma(u(q_b, q_b^g) - (1 + i_l^g)l\phi - \Gamma) \tag{A12}$$

where λ_L is the Lagrange multiplier on the constraint on a bank's lending. Also, λ_Γ is the Lagrange multiplier on the participation constraint of the borrower. The bank's first order condition for how much to loan becomes:

$$i + f - (1-f)i_d - \lambda_L + \lambda_\Gamma\Big[u_{q_b}(q_b, q_b^g)\frac{dq_b}{dl} - (1 + i_l)\phi\Big] = 0 \tag{A13}$$

Similar to what was described in BCW, with $i_l + f - (1-f)i_d > 0$ the bank will loan as much as possible. Therefore, $u_{q_b}(q_b, q_b^g)\frac{dq_b}{dl} = (1 + i_l)\phi$ and it will be the case that $\lambda_\Gamma > 0$. However, competition will drive interest rates to parity: $i_l = (1-f)i_d - f$.

From $q_b = \big((1-\psi)m_g + l\big)/p$

$$\frac{dq_b}{dl} = \frac{1}{p} \Rightarrow \frac{dl}{dq_b} = p \tag{A14}$$

As p^g increases, it costs more to obtain goods in the DM and the buyer will choose to borrow more funds. Since $p\phi = c_{1,q_s}(q_s, q_s^g)$

$$\frac{dq_b}{dl} = \frac{\phi}{c_{1,q_s}(q_s, q_s^g)} \tag{A15}$$

Thus, a bank's decision on how much to lend becomes

$$i + f - (1-f)i_d - \lambda_L + \lambda_\Gamma\Big[u_{q_b}(q_b, q_b^g)\frac{\phi^g}{c_{1,q_s}(q_s, q_s^g)} - (1 + i)\phi\Big] = 0 \tag{A16}$$

Since the bank will receive no profits due to free entry:

$$\phi\lambda_\Gamma\Big[\frac{u_{q_b}(q_b,q_b^g)}{c_{1,q_s^g}(q_s,q_s^g)}-(1+i)\Big]=\lambda_L \tag{A17}$$

$$\frac{u_{q_b^g}(q_b,q_b^g)}{c_{1,q_s^g}(q_s,q_s^g)}=1+i+\frac{\lambda_L}{\phi\lambda_\Gamma} \tag{A18}$$

We can represent the bank's problem for government money as

$$\mathscr{L}\max_{l_g}=(i_l^g+f-(1-f)i_d)l_g-\lambda_L(l-\bar{l})+\lambda_\Gamma(u(q_b,q_b^g)-(1+i_l^g)l\phi^g-\Gamma) \tag{A19}$$

where λ_L is the Lagrange multiplier on the constraint on a bank's lending. Also, λ_Γ is the Lagrange multiplier on the participation constraint of the borrower. The bank's first order condition for how much to loan becomes:

$$i_l^g+f-(1-f)i_d-\lambda_L+\lambda_\Gamma\Big[u_{q_b^g}(q_b,q_b^g)\frac{dq_b^g}{dl_g}-(1+i_l^g)\phi^g\Big]=0 \tag{A20}$$

Similar to what was described in BCW, with $i_l^g+f-(1-f)i_d^g>0$ the bank will loan as much as possible. Therefore, $u_{q_b^g}(q_b,q_b^g)\frac{dq_b^g}{dl_g}=(1+i_l^g)\phi$ and it will be the case that $\lambda_\Gamma>0$. However, competition will drive interest rates to parity: $i_l^g=(1-f)i_d^g-f$.

From $q_b^g=\big((1-\psi^g)m_g+l_g\big)/p^g$

$$\frac{dq_b^g}{dl_g}=\frac{1}{p^g}\Rightarrow\frac{dl_g}{dq_b^g}=p^g \tag{A21}$$

As p^g increases, it costs more to obtain goods in the DM and the buyer will choose to borrow more funds. Since $p^g\phi^g(1+i^g)=c_{1,q_s^g}(q_s,q_s^g)$

$$\frac{dq_b}{dl_g}=\frac{(1+i^g)\phi^g}{c_{1,q_s^g}(q_s,q_s^g)} \tag{A22}$$

Thus, a bank's decision on how much to lend becomes

$$i_l^g + f - (1-f)i_d^g - \lambda_L + \lambda_\Gamma \Big[u_{q_b^g}(q_b, q_b^g) \frac{(1+i^g)\phi^g}{c_{1,q_s^g}(q_s, q_s^g)} - (1+i_l^g)\phi^g \Big] = 0 \tag{A23}$$

Since the bank will receive no profits due to free entry:

$$\phi^g \lambda_\Gamma \Big[\frac{(1+i^g) u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} - (1+i_l^g) \Big] = \lambda_L \tag{A24}$$

$$\frac{(1+i^g) u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} = 1 + i_l^g + \frac{\lambda_L}{\phi \lambda_\Gamma} \tag{A25}$$

■

A5. Proof of Type III Bank's Problem

We can represent the bank's problem for private money as

$$\mathscr{L} \max_l = (i_l + f - (1-f)i_d) l_g - \lambda_L (l - \bar{l}) + \lambda_\Gamma (u(q_b, q_b^g) - (1+i_l^g) l\phi - \Gamma) \tag{A26}$$

where λ_L is the Lagrange multiplier on the constraint on a bank's lending. Also, λ_Γ is the Lagrange multiplier on the participation constraint of the borrower. The bank's first order condition for how much to loan becomes:

$$i + f - (1-f)i_d - \lambda_L + \lambda_\Gamma \Big[u_{q_b}(q_b, q_b^g) \frac{dq_b}{dl} - (1+i_l)\phi \Big] = 0 \tag{A27}$$

Similar to what was described in BCW, with $i_l + f - (1-f)i_d > 0$ the bank will loan as much as possible. Therefore, $u_{q_b}(q_b, q_b^g)\frac{dq_b}{dl} = (1+i_l)\phi$ and it will be the case that $\lambda_\Gamma > 0$. However, competition will drive interest rates to parity: $i_l = (1-f)i_d - f$.

From $q_b = \big(m_g + l\big)/p$

$$\frac{dq_b}{dl} = \frac{1}{p} \Rightarrow \frac{dl}{dq_b} = p \tag{A28}$$

As p^g increases, it costs more to obtain goods in the DM and the buyer will choose to borrow more funds. Since $p\phi = c_{1,q_s}(q_s, q_s^g)$

$$\frac{dq_b}{dl} = \frac{\phi}{c_{1,q_s}(q_s, q_s^g)} \tag{A29}$$

Thus, a bank's decision on how much to lend becomes

$$i + f - (1-f)i_d - \lambda_L + \lambda_\Gamma \Big[u_{q_b}(q_b, q_b^g) \frac{\phi^g}{c_{1,q_s}(q_s, q_s^g)} - (1+i)\phi \Big] = 0 \tag{A30}$$

Since the bank will receive no profits due to free entry:

$$\phi \lambda_\Gamma \Big[\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} - (1+i) \Big] = \lambda_L \tag{A31}$$

$$\frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} = 1 + i + \frac{\lambda_L}{\phi \lambda_\Gamma} \tag{A32}$$

We can represent the bank's problem for government money as

$$\mathscr{L} \max_{l_g} = (i_l^g + f - (1-f)i_d) l_g - \lambda_L (l - \bar{l}) + \lambda_\Gamma (u(q_b, q_b^g) - (1 + i_l^g) l \phi^g - \Gamma) \tag{A33}$$

where λ_L is the Lagrange multiplier on the constraint on a bank's lending. Also, λ_Γ is the Lagrange multiplier on the participation constraint of the borrower. The bank's first order condition for how much to loan becomes:

$$i_l^g + f - (1-f)i_d - \lambda_L + \lambda_\Gamma \Big[u_{q_b^g}(q_b, q_b^g) \frac{dq_b^g}{dl_g} - (1 + i_l^g)\phi^g \Big] = 0 \tag{A34}$$

Similar to what was described in BCW, with $i_l^g + f - (1-f)i_d^g > 0$ the bank will loan as much as possible. Therefore, $u_{q_b^g}(q_b, q_b^g) \frac{dq_b^g}{dl_g} = (1 + i_l^g)\phi$ and it will be the case that $\lambda_\Gamma > 0$. However, competition will drive interest rates to parity: $i_l^g = (1-f)i_d^g - f$.

From $q_b^g = \big((m_g + l_g\big)/p^g$

$$\frac{dq_b^g}{dl_g} = \frac{1}{p^g} \Rightarrow \frac{dl_g}{dq_b^g} = p^g \tag{A35}$$

As p^g increases, it costs more to obtain goods in the DM and the buyer will choose to borrow more funds. Since $p^g\phi^g(1+i^g) = c_{1,q_s^g}(q_s, q_s^g)$

$$\frac{dq_b}{dl_g} = \frac{(1+i^g)\phi^g}{c_{1,q_s^g}(q_s, q_s^g)} \tag{A36}$$

Thus, a bank's decision on how much to lend becomes

$$i_l^g + f - (1-f)i_d^g - \lambda_L + \lambda_\Gamma\Big[u_{q_b^g}(q_b, q_b^g)\frac{(1+i^g)\phi^g}{c_{1,q_s^g}(q_s, q_s^g)} - (1+i_l^g)\phi^g\Big] = 0 \tag{A37}$$

Since the bank will receive no profits due to free entry:

$$\phi^g\lambda_\Gamma\Big[\frac{(1+i^g)u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} - (1+i_l^g)\Big] = \lambda_L \tag{A38}$$

$$\frac{(1+i^g)u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} = 1 + i_l^g + \frac{\lambda_L}{\phi\lambda_\Gamma} \tag{A39}$$

■

A6. Proof for Marginal Value of Money with Private Money with type I banks

The marginal value of digital money starts with:

$$V_m(m, m^g) = (1-n)\left[u_{q_b}(q_b, q_b^g)\frac{\partial q_b}{\partial m} + W_m\Big((1-\psi) - p\frac{\partial q_b}{\partial m} + (1-\psi)\frac{\partial l}{\partial m}\Big) + W_l\frac{\partial l}{\partial m}\right]$$

$$+n\left[-c_{1,q_s}(q_s, q_s^g)\frac{\partial q_s}{\partial m} + W_m\left((1-\psi) + p\frac{\partial q_s}{\partial m} - (1-\psi)\frac{\partial d}{\partial m}\right) + W_d\frac{\partial d}{\partial m}\right] \tag{A40}$$

Money brought into the DM by a seller will not affect his production of the DM good. Therefore, we can substitute $\partial q_s/\partial m = 0$ for the consumer's decision on money holdings.

Also, with $i_d > -\psi$, a seller will deposit all money holdings and we need to substitute $\partial d/\partial m = 1$.

Lastly, use the envelope conditions for cash, loans and deposits to get

$$V_m(m, m^g) = (1-n)\left[u_{q_b}(q_b, q_b^g)\frac{\partial q_b}{\partial m} + \phi\Big((1-\psi) - p\frac{\partial q_b}{\partial m} + (1-\psi)\frac{\partial l}{\partial m}\Big) - \phi(1+i)\frac{\partial l}{\partial m}\right] + n\phi(1+i_d) \tag{A41}$$

If a buyer in the first market, the consumer must weigh the additional utility provided from consumption of the first market good bought with an additional unit of money. Additionally, the buyer considers the gain in utility from borrowing less funds from banks. If a seller, an additional unit of money will reduce his labor in the second market. In particular, this will depend on the interest rate paid by banks on deposits of tokens.

All of the tokens held by a buyer will be spent in the first market. Consequently, $p(\partial q_b/\partial m) = (1-\psi) + (1-\psi)\partial l/\partial m$ and the first order condition becomes:

$$V_m(m, m^g) = (1-n)\left[u_{q_b}(q_b, q_b^g)\frac{\partial q_b}{\partial m} - \phi(1+i)\frac{\partial l}{\partial m}\right] + n\phi(1+i_d) \tag{A42}$$

As a buyer, the net role of money demand is to purchase more goods and reduce borrowing costs. Now we can substitute $\frac{\partial l}{\partial m} = \frac{p}{(1-\psi)}\frac{\partial q_b}{\partial m} - 1$

$$V_m(m, m^g) = (1-n)\left[\frac{\partial q_b}{\partial m}(u_{q_b}(q_b, q_b^g) - \frac{\phi(1+i)p}{(1-\psi)}) + \phi(1+i)\right] + n\phi(1+i_d) \tag{A43}$$

Recall the equilibrium conditions from the credit market, $\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s)} = \frac{(1+i)}{(1-\psi)}$. Next, substitute

$c_{1,q_s}(q_s, q_s^g) = p\phi$. Then, we can apply $u_{q_b}(q_b, q_b^g) = p\phi\frac{(1+i)}{(1-\psi)}$ to obtain

$$V_m(m, m^g) = (1-n)\frac{(1-\psi)u_{q_b}(q_b, q_b^g)}{p} + n\phi(1+i_d) \tag{A44}$$

In a symmetric equilibrium $q_s = \frac{1-n}{n}q_b$. If $m < m^*$ then $0 < q_b < q^*$ and so $\partial q_b/\partial m > 0$. Thus, $V_{mm}(m, m^g) < 0$. When $m \geq m^*$ then $q_b = q^*$ implying $\partial q_b/\partial m = 0$. Thus, $V_{mm}(m, m^g) = 0$. Hence, $V(m, m^g)$ is concave in m.

Substitute $c_{1,q_s}(q_s, q_s^g) = p\phi$ to get

$$V_m(m, m^g) = \phi\left[(1-n)\frac{(1-\psi)u_{q_b}(q_b, q_b^g)}{c_{q_s}(q_s, q_s^g)} + n(1+i_d)\right] \tag{A45}$$

■

A7. Proof of the marginal value of government money with Type I Banks

The marginal value of government money starts with:

$$V_{m^g}(m, m^g) = (1-n)\left[u_{q_b^g}(q_b, q_b^g)\frac{\partial q_b^g}{\partial m^g} + W_{m,g}\Big((1-\psi^g) - p^g\frac{\partial q_b^g}{\partial m^g} + (1-\psi^g)\frac{\partial l_g}{\partial m^g}\Big) + W_l^g\frac{\partial l_g}{\partial m^g}\right]$$

$$+n\left[-c_{1,q_s^g}(q_s, q_s^g)\frac{\partial q_s^g}{\partial m^g} + W_{m,g}\left((1-\psi^g) + p^g\frac{\partial q_s^g}{\partial m} - (1-\psi^g)\frac{\partial d_g}{\partial m^g} - (1-\psi^g)\frac{\partial d_{c,g}}{\partial m^g}\right) + W_{d,g}\frac{\partial d_g}{\partial m^g} + W_{d,e}\frac{\partial d_{c,g}}{\partial m^g}\right] \tag{A46}$$

Money brought into the DM by a seller will not affect his production of the DM good. Therefore, we can substitute $\partial q_s^g/\partial m^g = 0$ for the consumer's decision on money holdings. We need to substitute $\partial d^g/\partial m^g = 1$ and $\partial d^{c,g}/\partial m^g = 0$. This gives us

$$V_{m^g}(m, m^g) = (1-n)\left[u_{q_b^g}(q_b, q_b^g)\frac{\partial q_b^g}{\partial m^g} + W_{m,g}\Big((1-\psi^g) - p^g\frac{\partial q_b^g}{\partial m^g} + (1-\psi^g)\frac{\partial l_g}{\partial m^g}\Big) + W_l^g\frac{\partial l_g}{\partial m^g}\right]$$

$$+n\left[-c_{1,q_s^g}(q_s,q_s^g)\cdot 0+W_{m,g}\left((1-\psi^g)+0-(1-\psi^g)\cdot 1-(1-\psi^g)\cdot 0\right)+W_{d,g}\cdot 1+W_{d,e}\cdot 0\right] \quad \text{(A47)}$$

Next, use the envelope conditions for government money, loans and deposits to get

$$V_{m^g}(m,m^g)=(1-n)\left[u_{q_b^g}(q_b,q_b^g)\frac{\partial q_b^g}{\partial m^g}+\phi^g(1+i^g)\Big((1-\psi^g)-p^g\frac{\partial q_b^g}{\partial m^g}+(1-\psi^g)\frac{\partial l_g}{\partial m^g}\Big)+\big(-\phi^g(1+i_l^g)\big)\frac{\partial l_g}{\partial m^g}\right]$$

$$+n\left[\phi^g(1+i^g)(1+i_d^g)(1-f)\right] \quad \text{(A48)}$$

If a buyer in the first market, the consumer must weigh the additional utility provided from consumption of the first market good bought with an additional unit of government money. Additionally, the buyer considers the gain in utility from borrowing less funds from banks. The funds borrowed are subject to loss as well. If a seller, an additional unit of money will reduce his labor in the second market. In particular, this will depend on the interest rate paid by banks on deposits of tokens.

All of the tokens held by a buyer will be spent in the first market. Consequently, $p^g(\partial q_b^g/\partial m^g)=(1-\psi^g)+(1-\psi^g)\partial l_g/\partial m^g$ and the first order condition becomes:

$$V_{m^g}(m,m^g)=(1-n)\left[u_{q_b^g}(q_b,q_b^g)\frac{\partial q_b^g}{\partial m^g}-\phi^g(1+i_l^g)\frac{\partial l_g}{\partial m^g}\right]+n\phi^g(1+i^g)(1+i_d^g)(1-f) \quad \text{(A49)}$$

As a buyer, the net role of money demand is to purchase more goods and reduce borrowing costs. Now we can substitute $(1-\psi^g)\frac{\partial l_g}{\partial m^g}=p^g\frac{\partial q_b^g}{\partial m^g}-(1-\psi^g)$

$$V_{m^g}(m,m^g)=(1-n)\left[\frac{\partial q_b^g}{\partial m^g}\big(u_{q_b^g}(q_b,q_b^g)-\frac{p^g\phi^g(1+i_l^g)}{(1-\psi^g)}\big)+\phi^g(1+i_l^g)\right]+n\phi^g(1+i^g)(1+i_d^g)(1-f) \quad \text{(A50)}$$

Recall the equilibrium conditions from the credit market, $\frac{u_{q_b^g}(q_b,q_b^g)}{c_{1,q_s^g}(q_s,q_s^g)}=\frac{(1+i_l^g)}{(1+i^g)(1-\psi^g)}$. Next, substitute $c_{1,q_s^g}(q_s,q_s^g)=p^g\phi^g(1+i^g)$. Then, we can apply $u_{q_b^g}(q_b,q_b^g)=\frac{p^g\phi^g(1+i_l^g)}{(1-\psi^g)}$ to obtain

$$V_m^g(m, m^g) = (1-n)\left[\frac{(1-\psi^g)u_{q_b^g}(q_b, q_b^g)}{p^g}\right] + n\phi^g(1+i^g)(1+i_d^g)(1-f) \tag{A51}$$

■

A8. Proof of private money equilibrium conditions w/ type I banks

To find a monetary equilibrium with price stability, we start with the nominal interest rate that will clear the market. From $\phi = \beta V_{+1,m}(m_{+1}, m_{+1}^g)$, substitute into $(A45)$ to get:

$$\frac{\phi_{-1}}{\beta} = \phi\left[(1-n)\frac{(1-\psi)u_{q_b}(q_b, q_b^g)}{c_{q_s}(q_s, q_s^g)} + n(1+i_d)\right] \tag{A52}$$

Now use $q_s = \frac{1-n}{n}q_b$ and $\frac{\phi_{-1}}{\phi} = \gamma$ and subtract both sides by the real value of a token ϕ:

$$\frac{\gamma-\beta}{\beta} = (1-n)\left[\frac{(1-\psi)u_{q_b}(q_b, q_b^g)}{c_{q_s}(q_s, q_s^g)} - 1\right] + ni_d \tag{A53}$$

Since banks can enforce repayment, they will be unconstrained and $\bar{l} = \infty$. Consequently, $\lambda_l = 0$ and $\frac{u_{q_b}(q_b, q_b^g)}{c_{q_s}(q_s, q_s^g)} = 1 + i$ in (A53) yields

$$\frac{\gamma-\beta}{\beta} = (1-n)((1-\psi)(1+i) - 1) + ni_d \tag{A54}$$

Zero profits for banks implies $i = (1-f)i_d - f$. Therefore, we have

$$\frac{\gamma-\beta}{\beta} = i \tag{A55}$$

In terms of q_b:

$$\frac{\gamma-\beta}{\beta} = \frac{(1-\psi)u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} - 1 \tag{A56}$$

■

A9. Proof of government money equilibrium conditions w/ type I banks

We now find the monetary equilibrium with credit using government money. Akin to the derivation for private money, we ultimately obtain:

$$\frac{\phi^g_{-1}}{\beta} = \phi^g \left[(1-n) \frac{(1-\psi^g)(1+i^g) u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} + n(1+i^g)(1+i_d^g)(1-f) \right] \tag{A57}$$

Next substitute $q_s^g = \frac{1-n}{n} q_b^g$ and $\frac{\phi^g_{-1}}{\phi^g} = \gamma^g$ and subtract both sides by the real value of a token ϕ:

$$\frac{\gamma^g - \beta}{\beta} = (1-n) \frac{(1-\psi^g)(1+i^g) u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} + n(1+i^g)(1+i_d^g)(1-f) - 1 \tag{A58}$$

The right-hand side measures the value of bringing a marginal unit of money into the first market. The first term is lowered by the potential to lose currency in the first market. Additionally, the liquidity benefit of money in the DM increases with the interest rate paid out on CBDC. The seller values CBDC more at a higher interest rate and will produce more goods as a result.

Since banks can enforce repayment, they will be unconstrained and $\bar{l}_g = \infty$. Consequently, $\lambda_l^g = 0$ and $\frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} = \frac{1+i_l^g}{(1+i^g)(1-\psi^g)}$ in (A58) yields

$$\frac{\gamma^g - \beta}{\beta} = (1-n)(1+i_l^g) + n(1+i^g)(1+i_d^g)(1-f) - 1 \tag{A59}$$

The first term now represents the interest saving from borrowing one less unit of money when a buyer.

Zero profits for banks implies $1 + i_l^g = (1+i_d^g)(1-f) = 1 + i_{l,d}^g$:

$$\frac{\gamma^g - \beta(1+ni^g)}{\beta(1+ni^g)(1-f)} + \frac{f}{1-f} = i_d^g \tag{A60}$$

Next, q_b^g is pinned down through the following condition:

$$\frac{\gamma^g - \beta(1+ni^g)}{\beta(1+ni^g)} = \frac{(1+i^g)(1-\psi^g)u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} - 1 \tag{A61}$$

■

A10. Proof of Physical Currency driving out Private Tokens from Circulation w/ type I banks

To begin, divide (A56) by $(1-\psi)$

$$\frac{\gamma - \beta}{\beta(1-\psi)} = \frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} - \frac{1}{1-\psi} \tag{A62}$$

This reduces to

$$\frac{\gamma}{\beta(1-\psi)} = \frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, \frac{1-n}{n}q_b^g)} \tag{A63}$$

Now, we set $\psi^g = 0$ and $i^g = 0$ for (A61) and then add 1 to both sides

$$\frac{\gamma^g}{\beta} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, \frac{1-n}{n}q_b^g)} \tag{A64}$$

As the right hand side goes up trade of the good goes down. In order for physical government money to dominate, we need

$$\frac{\gamma^g}{\beta} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, \frac{1-n}{n}q_b^g)} < \frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, \frac{1-n}{n}q_b^g)} = \frac{\gamma}{\beta(1-\psi)} \tag{A65}$$

which simplifies down to

$$\gamma^g < \frac{\gamma}{(1-\psi)} \tag{A66}$$

We can substitute $\gamma^g = 1 + \sigma^g$ and set the private growth rate with no cap $\gamma = 1$. We can then solve for growth rate of government money where physical currency dominates.

$$\sigma^g < \frac{\psi}{(1-\psi)} \tag{A67}$$

■

A11. Proof of Private Tokens driving out CBDC w/ type I banks

We begin by dividing (A61) by $(1-\psi^g)(1+i^g)$

$$\frac{\gamma^g - \beta(1+ni^g)}{\beta(1+ni^g)(1-\psi^g)(1+i^g)} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} - \frac{1}{(1-\psi^g)(1+i^g)} \tag{A68}$$

which simplifies to

$$\frac{\gamma^g}{\beta(1+ni^g)(1-\psi^g)(1+i^g)} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} \tag{A69}$$

In order for private money to drive out the CBDC, we need

$$\frac{\gamma^g}{\beta(1+ni^g)(1-\psi^g)(1+i^g)} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, \frac{1-n}{n}q_b^g)} > \frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, \frac{1-n}{n}q_b^g)} = \frac{\gamma}{\beta(1-\psi)} \tag{A70}$$

This reduces to

$$\frac{\gamma^g}{(1+ni^g)(1-\psi^g)(1+i^g)} > \frac{\gamma}{(1-\psi)} \tag{A71}$$

Without a cap on private token production $\gamma = 1$ and we can assume $\psi = \psi^g$ if the security technology is the same for both types of currency. The inequality becomes

$$\frac{\gamma^g}{(1+ni^g)(1+i^g)} > 1 \tag{A72}$$

If we substitute $\gamma^g = 1 + \sigma^g$ and simplify we have

$$\sigma^g > n(i^g)^2 + (1+n)i^g \tag{A73}$$

■

A12. Proof of Private Token Production w/ Type II Banks

First, suppose that $\phi = \overline{\phi}$, then token production will be equal to what is needed for a

stable price $\Delta^* = \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. Since token production is equal to the amount of tokens lost it will be the case that $\gamma = 1$ and thus the price level will be constant over time. The price of tokens will be the same as the previous period $\phi_{+1} = \phi$. This will lead to the same amount of tokens being created as the previous period $\Delta^*_{+1} = \Delta^*$.

This process will repeat itself ensuring token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = (1-n)\psi\overline{M}$. Thus, $\gamma = 1$ for the remaining periods making a steady state possible.

Next, assume $\phi > \overline{\phi}$, then token production will greater than what is needed for a stable price $\Delta^* > \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. Since token production will be greater than the amount of tokens lost it will be the case that $\gamma > 1$ and thus inflation occurs. Over time, the price of tokens will be less than the previous period $\phi_{+1} < \phi$. This will lead to less tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} < \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = (1-n)\psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with inflation in this economy.

Lastly, suppose $\phi < \overline{\phi}$, then token production is less than what is needed for a stable price $\Delta^* < \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. There will be less new tokens created than tokens lost from the money supply in the previous period $\Delta^* < (1-n)\psi M_{-1}$. Since token production will be less than the amount of tokens lost it will be the case that $\gamma < 1$ and thus deflation is observed. The price of tokens will be greater than the previous period $\phi_{+1} > \phi$. This will lead to more tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} > \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with deflation in this economy. ■

A13. Proof of Type II buyer's problem

The buyer's first order condition for choosing how much of goods to purchase through government money is

$$u_{q_b^g}(q_b, q_b^g) = p^g W_{m^g} + p^g \lambda^g. \tag{A74}$$

Akin to the use of digital tokens, the buyer's desired amount of consumption depends on the cost of working to repay the loan and whether the liquidity constraint binds. Upon substituting $W_{m,g} = \phi^g(1 + i^g)$ and $\frac{c_{1,q_s^g}(q_s, q_s^g)}{\phi^g} = p^g(1 + i^g)$

$$u_{q_b^g}(q_b, q_b^g) = c_{1,q_s^g}(q_s, q_s^g)\Big(1 + \frac{\lambda^g}{\phi^g(1 + i^g)}\Big) \tag{A75}$$

Similar to when private digital tokens are in use, the last equation compares the gains from additional consumption at the cost of producing goods. However, the seller will consider the interest paid by the central bank when making their production decision. Additionally, the cash constraint will cause an inefficiency in trade.

The buyer's first order condition for l is

$$W_m + \lambda = -W_l + \lambda_l \tag{A76}$$

With the addition of type II banks, a buyer will value each unit of loans as cash at a higher rate W_m rather than $(1 - \psi)W_m$. Similarly, the buyer values the relief to the constraint on money holdings at a higher rate λ rather than $(1 - \psi)\lambda$. Both of these results are due to the bank's safekeeping. This will lead to the buyer taking out a larger size loan.

Substituting $W_m = \phi$ and $W_l = -\phi(1 + i)$ the previous envelope conditions, gives us $\lambda = \phi i + \lambda_l$. Upon substituting for λ, the quantity traded in the first market satisfies:

$$\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = 1 + i + \frac{\lambda_l}{\phi} \tag{A77}$$

In contrast to the setting with only type I banks, the rate of currency loss will play no direct

role in the loan decision. That is, through type II banks, the buyer will take out a larger loan at similar interest rates. From this perspective, the additional safe-keeping function of such intermediaries promotes credit market activity. If the borrowing constraint does not bind,

$$\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = 1 + i \tag{A78}$$

In this case the buyer will borrow up to the point where the marginal benefit of borrowing equals the marginal cost. He then spends all his money and consumes $q_b = ((1-\psi)m + l)/p$. However, trades are inefficient as long as $i > 0$. Unlike the case with type I banks, interest rates will never be negative.

The buyer's first order condition for l_g is

$$W_{m,g} + \lambda^g \leq -W_{l^g} + \lambda_l^g \tag{A79}$$

Identical to the loan decision of private money, the rate of currency loss will play no direct role in the loan decision. From the previous envelope conditions: $W_{m,g} = \phi^g(1 + i^g)$ and $W_{l^g} = -\phi^g(1 + i_l^g)$

$$\phi^g(1 + i^g) + \lambda^g \leq \phi^g(1 + i_l^g) + \lambda_l^g \tag{A80}$$

Upon substituting for λ^g, the quantity traded in the first market satisfies:

$$\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = \frac{\phi^g(1 + i_l^g)}{\phi^g(1 + i^g)} + \frac{\lambda_l^g}{\phi^g(1 + i^g)} \tag{A81}$$

If the borrowing constraint does not bind, $\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = \frac{(1+i_l^g)}{(1+i^g)}$. The buyer borrows up to the point where the marginal benefit of borrowing equals the marginal cost. He then spends all his government money and consumes $q_b^g = \big((1-\psi^g)m^g + l\big)/p^g$.

■

A14. Proof of Marginal Value of Government Money w/ type II banks

The marginal value of government money starts with:

$$V_{m^g}(m,m^g)=(1-n)\left[u_{q_b^g}(q_b,q_b^g)\frac{\partial q_b^g}{\partial m^g}+W_{m,g}\Big((1-\psi^g)-p^g\frac{\partial q_b^g}{\partial m^g}+\frac{\partial l_g}{\partial m^g}\Big)+W_l^g\frac{\partial l_g}{\partial m^g}\right] \quad \text{(A82)}$$

$$+n\left[-c_{1,q_s^g}(q_s,q_s^g)\frac{\partial q_s^g}{\partial m^g}+W_{m,g}\left((1-\psi^g)+p^g\frac{\partial q_s^g}{\partial m}-(1-\psi^g)\frac{\partial d_g}{\partial m^g}-(1-\psi^g)\frac{\partial d_{e,g}}{\partial m^g}\right)+W_{d,g}\frac{\partial d_g}{\partial m^g}+W_{d,e}\frac{\partial d_{e,g}}{\partial m^g}\right]$$

Money brought into the DM by a seller will not affect his production of the DM good. Therefore, we can substitute $\partial q_s^g/\partial m^g=0$ for the consumer's decision on money holdings. Let's assume conditions are such that the seller deposits funds with private banks. We need to substitute $\partial d^g/\partial m^g=1$ and $\partial d^{e,g}/\partial m^g=0$. This gives us

$$V_{m^g}(m,m^g)=(1-n)\left[u_{q_b^g}(q_b,q_b^g)\frac{\partial q_b^g}{\partial m^g}+W_{m,g}\Big((1-\psi^g)-p^g\frac{\partial q_b^g}{\partial m^g}+\frac{\partial l_g}{\partial m^g}\Big)+W_l^g\frac{\partial l_g}{\partial m^g}\right] \quad \text{(A83)}$$

$$+n\left[-c_{1,q_s^g}(q_s,q_s^g)\cdot 0+W_{m,g}\left((1-\psi^g)+0-(1-\psi^g)\cdot 1-(1-\psi^g)\cdot 0\right)+W_{d,g}\cdot 1+W_{d,e}\cdot 0\right]$$

Next, use the envelope conditions for government money, loans and deposits to get

$$V_{m^g}(m,m^g)=(1-n)\left[u_{q_b^g}(q_b,q_b^g)\frac{\partial q_b^g}{\partial m^g}+\phi^g(1+i^g)\Big((1-\psi^g)-p^g\frac{\partial q_b^g}{\partial m^g}+\frac{\partial l_g}{\partial m^g}\Big)+\big(-\phi^g(1+i_l^g)\big)\frac{\partial l_g}{\partial m^g}\right]$$

$$+n\left[\phi^g(1+i^g)(1+i_d^g)(1-f)\right] \quad \text{(A84)}$$

If a buyer in the first market, the consumer must weigh the additional utility provided from consumption of the first market good bought with an additional unit of money. Additionally, the buyer considers the gain in utility from borrowing less funds from banks. However, the funds borrowed are not subject to loss. If a seller, an additional unit of money will reduce his labor in the second market. In particular, this will depend on the interest rate paid by banks on deposits of tokens.

All of the tokens held by a buyer will be spent in the first market. Consequently,

$p^g(\partial q_b^g/\partial m^g) = (1-\psi^g) + \partial l_g/\partial m^g$ and the first order condition becomes:

$$V_{m^g}(m, m^g) = (1-n)\left[u_{q_b^g}(q_b, q_b^g)\frac{\partial q_b^g}{\partial m^g} - \phi^g(1+i_l^g)\frac{\partial l_g}{\partial m^g}\right] + n\phi^g(1+i^g)(1+i_d^g)(1-f) \quad \text{(A85)}$$

As a buyer, the net role of money demand is to purchase more goods and reduce borrowing costs. Now we can substitute $\frac{\partial l_g}{\partial m^g} = p^g\frac{\partial q_b^g}{\partial m^g} - (1-\psi^g)$

$$V_{m^g}(m, m^g) = (1-n)\left[\frac{\partial q_b^g}{\partial m^g}\big(u_{q_b^g}(q_b, q_b^g) - p^g\phi^g(1+i_l^g)\big) + \phi^g(1+i_l^g)(1-\psi^g)\right] + n\phi^g(1+i^g)(1+i_d^g)(1-f) \quad \text{(A86)}$$

Recall the equilibrium conditions from the credit market, $\frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} = \frac{(1+i_l^g)}{(1+i^g)}$. Next, substitute $c_{1,q_s^g}(q_s, q_s^g) = p^g\phi^g(1+i^g)$. Then, we can apply $u_{q_b^g}(q_b, q_b^g) = p^g\phi^g(1+i_l^g)$ to obtain

$$V_{m^g}(m, m^g) = (1-n)\left[\frac{\partial q_b^g}{\partial m^g}\big(u_{q_b^g}(q_b, q_b^g) - u_{q_b^g}(q_b, q_b^g)\big) + u_{q_b^g}(q_b, q_b^g)\frac{(1-\psi^g)}{p^g}\right] + n\phi^g(1+i^g)(1+i_d^g)(1-f) \quad \text{(A87)}$$

Upon simplifying:

$$V_{m^g}(m, m^g) = (1-n)\left[\frac{(1-\psi^g)u_{q_b^g}(q_b, q_b^g)}{p^g}\right] + n\phi^g(1+i^g)(1+i_d^g)(1-f) \quad \text{(A88)}$$

Substitute $c_{1,q_s^g}(q_s, q_s^g) = p^g\phi^g(1+i^g)$ to get

$$V_{m^g}(m, m^g) = \phi^g\left[(1-n)\frac{(1-\psi^g)(1+i^g)u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} + n(1+i^g)(1+i_d^g)(1-f)\right] \quad \text{(A89)}$$

■

A15. Proof of the Equilibrium Interest Rate for Government Money w/ Type II Banks

We find the monetary equilibrium with credit using government money. We look at the case when sellers deposit with private banks first. From $\phi^g = \beta V_{+1,m^g}(m_{+1}, m_{+1}^g)$, substitute

into $(A89)$ to get:

$$\frac{\phi^g_{-1}}{\beta} = \phi^g \left[(1-n) \frac{(1-\psi^g)(1+i^g) u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} + n(1+i^g)(1+i_d^g)(1-f) \right] \tag{A90}$$

Now use $q_s^g = \frac{1-n}{n} q_b^g$ and $\frac{\phi^g_{-1}}{\phi^g} = \gamma^g$ and subtract both sides by the real value of a token ϕ:

$$\frac{\gamma^g - \beta}{\beta} = (1-n) \frac{(1-\psi^g)(1+i^g) u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} + n(1+i^g)(1+i_d^g)(1-f) - 1 \tag{A91}$$

Since banks can enforce repayment, they will be unconstrained and $\bar{l}_g = \infty$. Consequently, $\lambda_l^g = 0$ and $\frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} = \frac{1+i_l^g}{1+i^g}$ in (A91) yields

$$\frac{\gamma^g - \beta}{\beta} = (1-n)(1-\psi^g)(1+i_l^g) + n(1+i^g)(1+i_d^g)(1-f) - 1 \tag{A92}$$

Zero profits for banks implies $1 + i_l^g = (1+i_d^g)(1-f) = 1 + i_{l,d}^g$:

$$\frac{\gamma^g - \beta(1 + ni^g - (1-n)\psi^g)}{\beta((1-n)(1-\psi^g) + n(1+i^g))} = i_{l,d}^g = i_l^g \tag{A93}$$

Find i_d^g

$$\frac{\gamma^g - \beta(1 + ni^g - (1-n)\psi^g)}{\beta((1-n)(1-\psi^g) + n(1+i^g)(1-f))} + \frac{f}{1-f} = i_d^g \tag{A94}$$

Now to get equilibrium quantity substitute into $\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = \frac{(1+i_l^g)}{(1+i^g)}$

$$\frac{(1+i^g) u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = 1 + \frac{\gamma^g - \beta(1 + ni^g - (1-n)\psi^g)}{\beta((1-n)(1-\psi^g) + n(1+i^g))} \tag{A95}$$

After simplifying:

$$\frac{\big((1-n)(1-\psi^g)(1+i^g) + n(1+i^g)^2\big) u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} - 1 = \frac{\gamma^g - \beta}{\beta} \tag{A96}$$

■

A16. Proof of Physical Currency driving out Private Tokens from Circulation w/ Type II Banks

To begin, divide (73) from the main text by $(1-(1-n)\psi)$

$$\frac{\gamma-\beta}{\beta(1-(1-n)\psi)}=\frac{u_{q_b}(q_b,q_b^g)}{c_{1,q_s}(q_s,q_s^g)}-\frac{1}{1-(1-n)\psi} \tag{A97}$$

This reduces to

$$\frac{\gamma}{\beta(1-(1-n)\psi)}=\frac{u_{q_b}(q_b,q_b^g)}{c_{1,q_s}(q_s,\frac{1-n}{n}q_b^g)} \tag{A98}$$

Now, we set $\psi^g=0$ and $i^g=0$ for (A96) and then add 1 to both sides

$$\frac{\gamma^g}{\beta}=\frac{u_{q_b^g}(q_b,q_b^g)}{c_{1,q_s^g}(q_s,\frac{1-n}{n}q_b^g)} \tag{A99}$$

As the right hand side goes up trade of the good goes down. In order for physical government money to dominate, we need

$$\frac{\gamma^g}{\beta}=\frac{u_{q_b^g}(q_b,q_b^g)}{c_{1,q_s^g}(q_s,\frac{1-n}{n}q_b^g)}<\frac{u_{q_b}(q_b,q_b^g)}{c_{1,q_s}(q_s,\frac{1-n}{n}q_b^g)}=\frac{\gamma}{\beta(1-(1-n)\psi)} \tag{A100}$$

which simplifies down to

$$\gamma^g<\frac{\gamma}{(1-(1-n)\psi)} \tag{A101}$$

We can substitute $\gamma^g=1+\sigma^g$ and set the private growth rate with no cap $\gamma=1$. We can then solve for growth rate of government money where physical currency dominates.

$$\sigma^g<\frac{(1-n)\psi}{(1-(1-n)\psi)} \tag{A102}$$

■

A17. Proof of Conditions for Private Money to drive out CBDC w/ Type II Banks

Take (A96) and add 1 and then divide by $\big((1-n)(1-\psi^g)(1+i^g)+n(1+i^g)^2\big)$

$$\frac{u_{q_b}(q_b,q_b^g)}{c_{1,q_s}(q_s,q_s^g)}=\frac{\gamma^g}{\beta\big((1-n)(1-\psi^g)(1+i^g)+n(1+i^g)^2\big)} \tag{A103}$$

Now, to show private money drive out CBDC we need

$$\frac{\gamma^g}{\beta\big((1-n)(1-\psi^g)(1+i^g)+n(1+i^g)^2\big)}=\frac{u_{q_b^g}(q_b,q_b^g)}{c_{1,q_s^g}(q_s,\frac{1-n}{n}q_b^g)}>\frac{u_{q_b}(q_b,q_b^g)}{c_{1,q_s}(q_s,\frac{1-n}{n}q_b^g)}=\frac{\gamma}{\beta(1-(1-n)\psi)} \tag{A104}$$

which reduces to

$$\gamma^g>\frac{\gamma\big((1-n)(1-\psi^g)(1+i^g)+n(1+i^g)^2\big)}{(1-(1-n)\psi)} \tag{A105}$$

As before, we set the gross growth rate for government money $\gamma^g=1+\sigma^g$ and set private growth rate so that $\gamma=1$. We can then solve for growth rate of government money where private money dominates.

$$1+\sigma^g>\frac{\big((1-n)(1-\psi^g)(1+i^g)+n(1+i^g)^2\big)}{(1-(1-n)\psi)} \tag{A106}$$

After simplifying

$$\sigma^g>\frac{\big((1-n)(1-\psi^g)(1+i^g)+n(1+i^g)^2\big)-(1-(1-n)\psi)}{(1-(1-n)\psi)} \tag{A107}$$

■

A18. Proof of Money Growth Threshold comparsion Type I Banks vs Type II Banks

We start by comparing

$$\sigma_I^g=n(i^g)^2+(1+n)i^g<\frac{\big((1-n)(1-\psi^g)(1+i^g)+n(1+i^g)^2\big)-(1-(1-n)\psi)}{(1-(1-n)\psi)}=\sigma_{II}^g \tag{A108}$$

Multiply both sides by $(1-(1-n)\psi)$ and assume the threshold is larger for type II banks

$$(n(i^g)^2+(1+n)i^g)((1-(1-n)\psi)) < \left((1-n)(1-\psi^g)(1+i^g)+n(1+i^g)^2\right)-(1-(1-n)\psi) \quad \text{(A109)}$$

After simplifying

$$-n(i^g)^2(1-n)\psi-(1+n)i^g(1-n)\psi < (1-n)-(1-n)\psi^g-(1-n)\psi^g i^g+n-(1-(1-n)\psi) \quad \text{(A110)}$$

Next, assume $\psi = \psi^g$

$$-n(i^g)^2(1-n)\psi - ni^g(1-n)\psi < 0 \quad \text{(A111)}$$

■

A19. Proof of Buyer's Loan Decision w/ Type III Banks

The buyer's first order condition for l_g is

$$W_{m,g} + \lambda^g \leq -W_{l^g} + \lambda_l^g \quad \text{(A112)}$$

Each unit of funding from the government money provides the buyer with more cash that he values at rate $W_{m,g}$. It also helps relieve the constraint on money holdings which he values at rate λ^g. However, the buyer will be held back by his debt which must be paid in units of the sovereign currency.

From the previous envelope conditions: $W_{m,g} = \phi^g(1+i^g)$ and $W_{l^g} = -\phi^g(1+i_l^g)$ we obtain

$$\lambda^g \leq \phi^g(1+i_l^g-(1+i^g)) + \lambda_l^g \quad \text{(A113)}$$

$$\lambda^g \leq \phi^g(i_l^g - i^g) + \lambda_l^g \quad \text{(A114)}$$

Upon substituting for λ^g, the quantity traded in the first market satisfies:

$$u_{q_b^g}(q_b, q_b^g) = c_{1,q_s^g}(q_s, q_s^g)\Big(1 + \frac{\phi^g(i_l^g - i^g) + \lambda_l^g}{\phi^g(1+i^g)}\Big) \tag{A115}$$

After simplifying

$$\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = \Big(\frac{\phi^g(1+i^g) + \phi^g(i_l^g - i^g) + \lambda_l^g}{\phi^g(1+i^g)}\Big)$$

$$\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = \frac{(1+i_l^g)}{(1+i^g)} + \frac{\lambda_l^g}{\phi^g(1+i^g)} \tag{A116}$$

If the borrowing constraint does not bind,

$$\frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, q_s^g)} = \frac{(1+i_l^g)}{(1+i^g)} \tag{A117}$$

■

A20. Proof of Marginal Value of Private Money w/ Type III Banks

Consumers can carry either government or a private digital currency into the DM. We will start by looking at the choice of private digital money holdings. In choosing the amount of private money balances to acquire in the CM, consumers need to consider the probability of being a buyer $(1-n)$ and being a seller (n) in the following DM. A buyer values their tokens for the first market good they can purchase. Also, if a buyer, they need to consider the amount of private digital money that can be borrowed from banks. This will reduce the amount of private digital tokens they carry into the first market. If a seller, the tokens derive worth from the store of value function. Token loss no longer plays a role in the analysis.

Thus, the marginal value of digital money starts with:

$$V_m(m, m^g) = (1-n)\left[u_{q_b}(q_b, q_b^g)\frac{\partial q_b}{\partial m} + W_m\Big(1 - p\frac{\partial q_b}{\partial m} + \frac{\partial l}{\partial m}\Big) + W_l\frac{\partial l}{\partial m}\right]$$

$$+n\left[-c_{1,q_s}(q_s,q_s^g)\frac{\partial q_s}{\partial m}+W_m\left(1+p\frac{\partial q_s}{\partial m}-\frac{\partial d}{\partial m}\right)+W_d\frac{\partial d}{\partial m}\right] \tag{A118}$$

Money brought into the DM by a seller will not affect his production of the DM good. Therefore, we can substitute $\partial q_s/\partial m = 0$ for the consumer's decision on money holdings. Also, with $i_d > 0$, a seller will deposit all money holdings and we need to substitute $\partial d/\partial m = 1$.

Lastly, use the envelope conditions for cash, loans and deposits to get

$$V_m(m,m^g) = (1-n)\left[u_{q_b}(q_b,q_b^g)\frac{\partial q_b}{\partial m}+\phi\Big(1-p\frac{\partial q_b}{\partial m}+\frac{\partial l}{\partial m}\Big)-\phi(1+i)\frac{\partial l}{\partial m}\right]+n\phi(1+i_d) \tag{A119}$$

If a buyer in the first market, the consumer must weigh the additional utility provided from consumption of the first market good bought with an additional unit of money. Additionally, the buyer considers the gain in utility from borrowing less funds from banks. If a seller, an additional unit of money will reduce his labor in the second market. In particular, this will depend on the interest rate paid by banks on deposits of tokens.

All of the tokens held by a buyer will be spent in the first market. Consequently, $p(\partial q_b/\partial m) = 1 + \partial l/\partial m$ and the first order condition becomes:

$$V_m(m,m^g) = (1-n)\left[u_{q_b}(q_b,q_b^g)\frac{\partial q_b}{\partial m}-\phi(1+i)\frac{\partial l}{\partial m}\right]+n\phi(1+i_d) \tag{A120}$$

As a buyer, the net role of money demand is to purchase more goods and reduce borrowing costs. Now we can substitute $\frac{\partial l}{\partial m} = p\frac{\partial q_b}{\partial m} - 1$

$$V_m(m,m^g) = (1-n)\left[\frac{\partial q_b}{\partial m}(u_{q_b}(q_b,q_b^g)-\phi(1+i)p)+\phi(1+i)\right]+n\phi(1+i_d) \tag{A121}$$

Recall the equilibrium conditions from the credit market, $\frac{u_{q_b}(q_b,q_b^g)}{c_{1,q_s}(q_s)} = 1+i$. Next, substitute $c_{1,q_s}(q_s,q_s^g) = p\phi$. Then, we can apply $u_{q_b}(q_b,q_b^g) = p\phi(1+i)$ to obtain

$$V_m(m,m^g) = (1-n)\frac{u_{q_b}(q_b,q_b^g)}{p}+n\phi(1+i_d) \tag{A122}$$

which is identical to BCW. In regards to the first term, an extra unit of money allows consumers to buy more goods. The second term reflects the store of value which is enhanced by depositing tokens with banks. In a symmetric equilibrium $q_s = \frac{1-n}{n} q_b$. If $m < m^*$ then $0 < q_b < q^*$ and so $\partial q_b / \partial m > 0$. Thus, $V_{mm}(m, m^g) < 0$. When $m \geq m^*$ then $q_b = q^*$ implying $\partial q_b / \partial m = 0$. Thus, $V_{mm}(m, m^g) = 0$. Hence, $V(m, m^g)$ is concave in m.

Substitute $c_{1,q_s}(q_s, q_s^g) = p\phi$ to get

$$V_m(m, m^g) = \phi \left[(1-n) \frac{(u_{q_b}(q_b, q_b^g)}{c_{q_s}(q_s, q_s^g)} + n(1+i_d) \right] \tag{A123}$$

Equation $(A123)$ reflects the dual role of money. When a buyer, he receives $u_{q_b}(q_b, q_b^g)/c_{1,q_s}(q_s, q_s^g)$ due to the medium of exchange role. If the agent is a seller, he lends money and receives $1 + i_d$ from the return on deposits of tokens. In particular, the extent of token loss reduces the value of money.

A21. Proof of Marginal Value of Government Money w/ Type III Banks

Now we look at the marginal value of government money. Here, sellers have the option to use a private or government bank. Lastly, the government money earns an interest which increase its store of value and purchasing power.

The marginal value of government money starts with:

$$V_{m^g}(m, m^g) = (1-n) \left[u_{q_b^g}(q_b, q_b^g) \frac{\partial q_b^g}{\partial m^g} + W_{m,g} \Big(1 - p^g \frac{\partial q_b^g}{\partial m^g} + \frac{\partial l_g}{\partial m^g} \Big) + W_l^g \frac{\partial l_g}{\partial m^g} \right]$$

$$+ n \left[- c_{1,q_s^g}(q_s, q_s^g) \frac{\partial q_s^g}{\partial m^g} + W_{m,g} \left(1 + p^g \frac{\partial q_s^g}{\partial m} - \frac{\partial d_g}{\partial m^g} - \frac{\partial d_{e,g}}{\partial m^g} \right) + W_{d,g} \frac{\partial d_g}{\partial m^g} + W_{d,e} \frac{\partial d_{e,g}}{\partial m^g} \right] \tag{A124}$$

Money brought into the DM by a seller will not affect his production of the DM good. Therefore, we can substitute $\partial q_s^g / \partial m^g = 0$ for the consumer's decision on money holdings. Let's assume conditions are such that the seller deposits funds with private banks. We need

to substitute $\partial d^g/\partial m^g = 1$ and $\partial d^{e,g}/\partial m^g = 0$. This gives us

$$V_{m^g}(m, m^g) = (1-n)\left[u_{q_b^g}(q_b, q_b^g)\frac{\partial q_b^g}{\partial m^g} + W_{m,g}\Big(1 - p^g\frac{\partial q_b^g}{\partial m^g} + \frac{\partial l_g}{\partial m^g}\Big) + W_l^g\frac{\partial l_g}{\partial m^g}\right]$$

$$+n\left[-c_{1,q_s^g}(q_s, q_s^g)\cdot 0 + W_{m,g}\left(1 + 0 - 1 - 0\right) + W_{d,g}\cdot 1 + W_{d,e}\cdot 0\right] \tag{A125}$$

Next, use the envelope conditions for government money, loans and deposits to get

$$V_{m^g}(m, m^g) = (1-n)\left[u_{q_b^g}(q_b, q_b^g)\frac{\partial q_b^g}{\partial m^g} + \phi^g(1+i^g)\Big(1 - p^g\frac{\partial q_b^g}{\partial m^g} + \frac{\partial l_g}{\partial m^g}\Big) + \big(-\phi^g(1+i_l^g)\big)\frac{\partial l_g}{\partial m^g}\right]$$

$$+n\left[\phi^g(1+i^g)(1+i_d^g)(1-f)\right] \tag{A126}$$

If a buyer in the first market, the consumer must weigh the additional utility provided from consumption of the first market good bought with an additional unit of money. Additionally, the buyer considers the gain in utility from borrowing less funds from banks. The funds borrowed are subject to loss as well. If a seller, an additional unit of money will reduce his labor in the second market. In particular, this will depend on the interest rate paid by banks on deposits of tokens.

All of the tokens held by a buyer will be spent in the first market. Consequently, $p^g(\partial q_b^g/\partial m^g) = 1 + \partial l_g/\partial m^g$ and the first order condition becomes:

$$V_{m^g}(m, m^g) = (1-n)\left[u_{q_b^g}(q_b, q_b^g)\frac{\partial q_b^g}{\partial m^g} - \phi^g(1+i_l^g)\frac{\partial l_g}{\partial m^g}\right] + n\phi^g(1+i^g)(1+i_d^g)(1-f) \tag{A127}$$

As a buyer, the net role of money demand is to purchase more goods and reduce borrowing costs. Now we can substitute $\frac{\partial l_g}{\partial m^g} = p^g\frac{\partial q_b^g}{\partial m^g} - 1$ to get

$$V_{m^g}(m, m^g) = (1-n)\left[\frac{\partial q_b^g}{\partial m^g}\big(u_{q_b^g}(q_b, q_b^g) - p^g\phi^g(1+i_l^g)\big) + \phi^g(1+i_l^g)\right] + n\phi^g(1+i^g)(1+i_d^g)(1-f) \tag{A128}$$

Recall the equilibrium conditions from the credit market, $\frac{u_{q_b^g}(q_b,q_b^g)}{c_{1,q_s^g}(q_s,q_s^g)} = \frac{(1+i_l^g)}{(1+i^g)}$. Next, substitute $c_{1,q_s^g}(q_s,q_s^g) = p^g\phi^g(1+i^g)$. Then, we can apply $u_{q_b^g}(q_b,q_b^g) = p^g\phi^g(1+i_l^g)$ to obtain

$$V_m^g(m,m^g) = (1-n)\left[\frac{u_{q_b^g}(q_b,q_b^g)}{p^g}\right] + n\phi^g(1+i^g)(1+i_d^g)(1-f) \tag{A129}$$

In regards to the first term, an extra unit of money allows consumers to buy more goods depending on the rate of token loss and prices in the DM. The second term reflects the store of value which is enhanced by depositing tokens with banks. These funds are subject to a service fee in addition to the interest earned. Additionally, all government money earns interest regardless of who holds it.

Substitute $c_{1,q_s^g}(q_s,q_s^g) = p^g\phi^g(1+i^g)$ to get

$$V_m^g(m,m^g) = \phi^g\left[(1-n)\frac{(1+i^g)u_{q_b^g}(q_b,q_b^g)}{c_{q_s^g}(q_s,q_s^g)} + n(1+i^g)(1+i_d^g)(1-f)\right] \tag{A130}$$

Equation $(A130)$ reflects the dual role of money. When a buyer, he receives $(1+i^g)u_{q_b^g}(q_b,q_b^g)/c_{1,q_s^g}(q_s,q_s^g)$ due to the medium of exchange role. If the agent is a seller, he lends money and receives $(1+i_d^g)(1-f)$ from the net return on deposits of tokens. In particular, the interest from the central bank increases the value of money.

A22. Proof of Equilibrium Conditions for Private Money w/ Type III Banks

In comparison to FS, we establish that a monetary equilibrium with price stability exists if the costs of producing new tokens are strictly convex where $c_2'(0) = 0$. To do so, we find the nominal interest rate that will clear the market. From $\phi = \beta V_{+1,m}(m_{+1}, m_{+1}^g)$, substitute into equation $(A45)$ to get:

$$\frac{\phi_{-1}}{\beta} = \phi\left[(1-n)\frac{u_{q_b}(q_b,q_b^g)}{c_{q_s}(q_s,q_s^g)} + n(1+i_d)\right] \tag{A131}$$

Now use $q_s = \frac{1-n}{n} q_b$ and $\frac{\phi_{-1}}{\phi} = \gamma$ and subtract both sides by the real value of a token ϕ:

$$\frac{\gamma - \beta}{\beta} = (1-n)\left[\frac{u_{q_b}(q_b, q_b^g)}{c_{q_s}(q_s, q_s^g)} - 1\right] + n i_d \tag{A132}$$

The right-hand side measures the value of bringing a marginal unit of money into the first market. The first term is the net benefit (marginal benefit minus marginal cost) of spending the unit of money on goods when a buyer and the second term is the net benefit from depositing a unit of idle balances when a seller.

Since banks can enforce repayment, they will be unconstrained and $\bar{l} = \infty$. Consequently, $\lambda_l = 0$ and $\frac{u_{q_b}(q_b, q_b^g)}{c_{q_s}(q_s, q_s^g)} = 1 + i$ in equation (A132) yields

$$\frac{\gamma - \beta}{\beta} = (1-n)i + n i_d \tag{A133}$$

The first term now represents the interest saving from borrowing one less unit of money when a buyer.

Zero profits for banks implies $i = i_d$:

$$\frac{\gamma - \beta}{\beta} = i \tag{A134}$$

In terms of q_b:

$$\frac{\gamma - \beta}{\beta} = \frac{u_{q_b}(q_b, q_b^g)}{c_{q_s}(q_s, q_s^g)} - 1 \tag{A135}$$

■

A23. Proof of Equilibrium for Government Money w/ Type III Banks

We now find the monetary equilibrium with credit using government money. We look at the case when sellers deposit with private banks first. From $\phi^g = \beta V_{+1,m^g}(m_{+1}, m_{+1}^g)$,

substitute into ($A130$) to get:

$$\frac{\phi_{-1}^g}{\beta} = \phi^g \left[(1-n) \frac{(1+i^g) u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} + n(1+i^g)(1+i_d^g)(1-f) \right] \tag{A136}$$

Now use $q_s^g = \frac{1-n}{n} q_b^g$ and $\frac{\phi_{-1}^g}{\phi^g} = \gamma^g$ and subtract both sides by the real value of a token ϕ:

$$\frac{\gamma^g - \beta}{\beta} = (1-n) \frac{(1+i^g) u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} + n(1+i^g)(1+i_d^g)(1-f) - 1 \tag{A137}$$

The right-hand side measures the value of bringing a marginal unit of money into the first market. The first term is the net benefit (marginal benefit minus marginal cost) of spending the unit of money on goods when a buyer and the second term is the net benefit from depositing a unit of idle balances when a seller. The first term increases with the interest paid out on a CBDC. The seller will value the CBDC more with a higher interest and produce more goods as a result.

Since banks can enforce repayment, they will be unconstrained and $\bar{l}_g = \infty$. Consequently, $\lambda_l^g = 0$ and $\frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, q_s^g)} = \frac{1+i_l^g}{(1+i^g)}$ in equation (A137) yields

$$\frac{\gamma^g - \beta}{\beta} = (1-n)(1+i_l^g) + n(1+i^g)(1+i_d^g)(1-f) - 1 \tag{A138}$$

The first term now represents the interest saving from borrowing one less unit of money when a buyer.

Zero profits for banks implies $1 + i_l^g = (1+i_d^g)(1-f) = 1 + i_{l,d}^g$:

$$\frac{\gamma^g - \beta(1+ni^g)}{\beta(1+ni^g)} = i_{l,d}^g = i_l^g \tag{A139}$$

Now find i_d^g by finding i_l^g in terms of i_d^g

$$i_l^g = i_d^g - f - i_d^g f \tag{A140}$$

Now substitute $i_l^g = (1-f)i_d^g - f$ to get the interest on deposits

$$\frac{\gamma^g - \beta(1+ni^g)}{\beta(1+ni^g)(1-f)} + \frac{f}{1-f} = i_d^g \tag{A141}$$

■

A24. Proof of Physical Currency driving out Private Tokens from Circulation w/ Type III Banks

To begin, simplify equation (A135) to

$$\frac{\gamma}{\beta} = \frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, \frac{1-n}{n}q_b^g)} \tag{A142}$$

Now, we set $i^g = 0$ for equation (88) from the main text and then add 1 to both sides

$$\frac{\gamma^g}{\beta} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, \frac{1-n}{n}q_b^g)} \tag{A143}$$

As the right hand side goes up trade of the good goes down. In order for physical government money to dominate, we need

$$\frac{\gamma^g}{\beta} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, \frac{1-n}{n}q_b^g)} < \frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, \frac{1-n}{n}q_b^g)} = \frac{\gamma}{\beta} \tag{A144}$$

which simplifies down to

$$\gamma^g < \gamma \tag{A145}$$

We can substitute $\gamma^g = 1 + \sigma^g$ and set the private growth rate with no cap $\gamma = 1$. We can then solve for growth rate of government money where physical currency dominates.

$$\sigma^g < 0 \tag{A146}$$

■

A25. Proof of Private Tokens driving out CBDC w/ Type III Banks

We begin by dividing equation (88) from the maing text by $(1 + i^g)$

$$\frac{\gamma^g - \beta(1 + ni^g)}{\beta(1 + ni^g)(1 + i^g)} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} - \frac{1}{(1 + i^g)} \tag{A147}$$

which simplifies to

$$\frac{\gamma^g}{\beta(1 + ni^g)(1 + i^g)} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{q_s^g}(q_s, q_s^g)} \tag{A148}$$

In order for private money to drive out the CBDC, we need

$$\frac{\gamma^g}{\beta(1 + ni^g)(1 + i^g)} = \frac{u_{q_b^g}(q_b, q_b^g)}{c_{1,q_s^g}(q_s, \frac{1-n}{n} q_b^g)} > \frac{u_{q_b}(q_b, q_b^g)}{c_{1,q_s}(q_s, \frac{1-n}{n} q_b^g)} = \frac{\gamma}{\beta} \tag{A149}$$

This reduces to

$$\frac{\gamma^g}{(1 + ni^g)(1 + i^g)} > \gamma \tag{A150}$$

Without a cap on private token production $\gamma = 1$. The inequality becomes

$$\frac{\gamma^g}{(1 + ni^g)(1 + i^g)} > 1 \tag{A151}$$

If we substitute $\gamma^g = 1 + \sigma^g$ and simplify we have

$$\sigma^g > n(i^g)^2 + (1 + n)i^g \tag{A152}$$

■